High Order Adaptive Mesh Refinement: *Interface boundary conditions*

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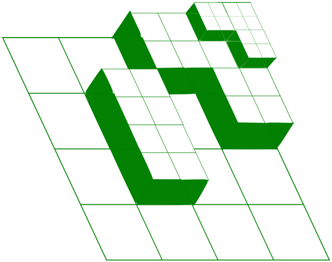
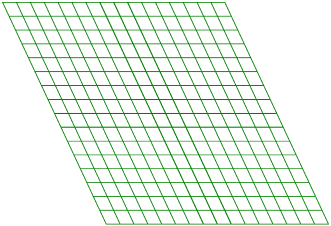
Motivation

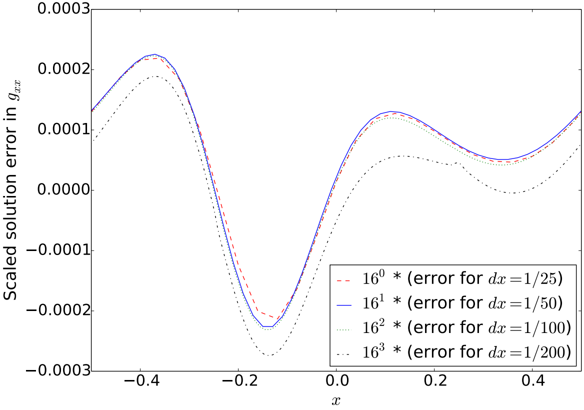
- High order methods (4th, 6th, etc) are becoming increasingly popular in Numerical relativity.
- In 3D simulations Mesh Refinement can lead to significant savings in computational resources.
- Current trends aim to combine the accuracy of high order methods with the adaptive resolution capabilities offered by AMR.
- Carpet, BAM, GridRipper, GRChombo, etc.

AMR Overview

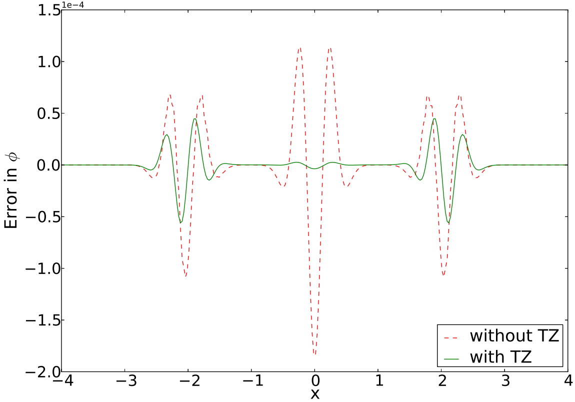




Challenge I: Order reduction



Challenge II: Spurious reflections



Other challenges

- Stability issues and loss of convergence at high resolutions.
- Spurious reflections at interface boundaries.
- Non trivial to code.
- Requires dynamic load balancing.
- Loss of conservation at interface boundaries.
- Robustness of error estimation methods.
- High order interpolation in three space dimensions is not cheap.

Continuous Runge Kutta Method

Intermediate stages

$$k_1 = hf(t, y)$$
$$k_2 = hf(t + \tfrac{1}{2}h, y + \tfrac{1}{2}hk_1)$$
$$k_3 = hf(t + \tfrac{1}{2}h, y + \tfrac{1}{2}hk_2)$$
$$k_4 = hf(t + h, y + k_3)$$

Avoid polynomial interpolation!

$$b_1(\theta) = \theta - \tfrac{3}{2}\theta^2 + \tfrac{2}{3}\theta^3$$
$$b_2(\theta) = \theta^2 - \tfrac{2}{3}\theta^3$$
$$b_3(\theta) = \theta^2 - \tfrac{2}{3}\theta^3$$
$$b_4(\theta) = -\tfrac{1}{2}\theta^2 + \tfrac{2}{3}\theta^3$$

$$y(t + \theta h) = y(t) + \sum_{i=1}^{4} b_i(\theta) k_i \qquad 0 \le \theta \le 1$$

$$\frac{d^{(m)}}{dt^{(m)}} y(t_n + \theta h) = \frac{1}{h^m} \sum_{i=1}^{s} k_i \frac{d^{(m)}}{d\theta^{(m)}} b_i(\theta) + \mathcal{O}(h^{4-m})$$

Boundary Conditions

- Step 1: Compute the Taylor expansions of the intermediate stages (Autonomous system).

$$
\begin{aligned}
k_1 &= hf \\
k_2 &= hf + \frac{h^2}{2} f_y f + \frac{h^3}{8} f_{yy}(f, f) + \frac{h^4}{48} f_{yyy}(f, f, f) + \mathcal{O}(h^5) \\
k_3 &= hf + \frac{h^2}{2} f_y f + \frac{h^3}{8} \left[f_{yy}(f, f) + 2 f_y f_y f \right] \\
&\quad + \frac{h^4}{48} \left[f_{yyy}(f, f, f) + 6 f_{yy}(f, f_y f) + 3 f_y f_{yy}(f, f) \right] + \mathcal{O}(h^5) \\
k_4 &= hf + h^2 f_y f + \frac{h^3}{2} \left[f_{yy}(f, f) + f_y f_y f \right] \\
&\quad + \frac{h^4}{24} \left[4 f_{yyy}(f, f, f) + 12 f_{yy}(f, f_y f) + 3 f_y f_{yy}(f, f) + 6 f_y f_y f_y f \right] + \mathcal{O}(h^5)
\end{aligned}
$$

Boundary Conditions

- Step 1: Compute the Taylor expansions of the intermediate stages (Autonomous system). One only needs the stages to third order accuracy in order ot achieve fourth order convergence!

$$k_1 = hf$$

$$k_2 = hf + \frac{h^2}{2} f_y f + \frac{h^3}{8} f_{yy}(f,f) + \frac{h^4}{48} f_{yyy}(f,f,f) + \mathcal{O}(h^5)$$

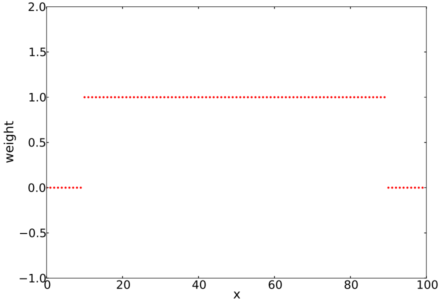
$$k_3 = hf + \frac{h^2}{2} f_y f + \frac{h^3}{8} \left[f_{yy}(f,f) + 2 f_y f_y f \right]$$

$$+ \frac{h^4}{48} \left[f_{yyy}(f,f,f) + 6 f_{yy}(f, f_y f) + 3 f_y f_{yy}(f,f) \right] + \mathcal{O}(h^5)$$

$$k_4 = hf + h^2 f_y f + \frac{h^3}{2} \left[f_{yy}(f,f) + f_y f_y f \right]$$

$$+ \frac{h^4}{24} \left[4 f_{yyy}(f,f,f) + 12 f_{yy}(f, f_y f) + 3 f_y f_{yy}(f,f) + 6 f_y f_y f_y f \right] + \mathcal{O}(h^5)$$

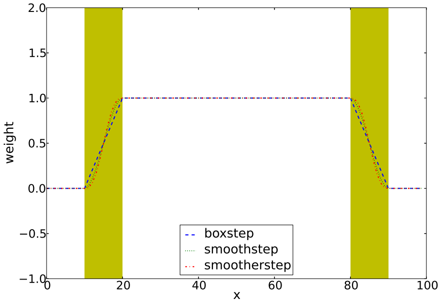
Boundary Conditions



- Step 2:
  Parameterize the solution in terms of a linear weight w as

$$g(x) = (1 - w)g(x, l_0) + wg(x, l_1)$$

Boundary Conditions



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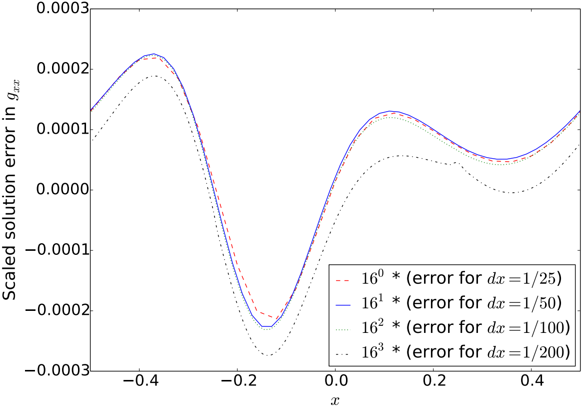
$$g(x) = (1 - w)g(x, l_0) + wg(x, l_1)$$

Smooth Transition Profiles

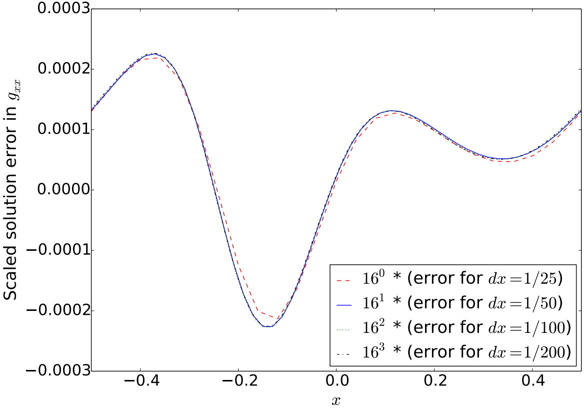
$$w(a, b, x) = t$$
$$w(a, b, x) = 3t^2 - 2t^3$$
$$w(a, b, x) = 10t^3 - 15t^4 + 6t^5$$

$$\text{where} \quad t = \begin{cases} 0 & \frac{x-a}{b-a} < 0 \\ 1 & \frac{x-a}{b-a} > 1 \\ \frac{x-a}{b-a} & \text{otherwise} \end{cases}$$

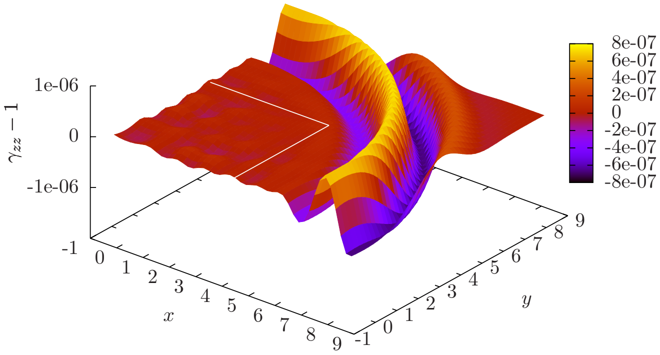
Example I: Gauge Wave



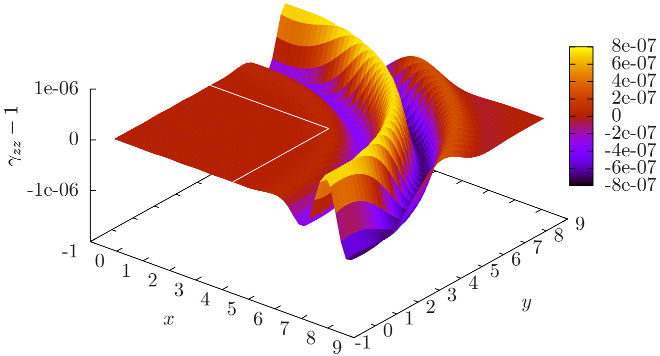
Example I: Gauge Wave



Example II: Teukolsky Wave



Example II: Teukolsky Wave



Summary

- Loss of convergence in high order AMR is caused by inconsistent application of boundary conditions, not stiffness.
- Spurious reflections are caused by differences in phase speeds, not round off error.